

# DISTINCT DISTANCES ON A SPHERE

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ABSTRACT. We prove that a set of  $N$  points on a two dimensional sphere satisfying a discrete energy condition determines at least a constant times  $N$  distinct distances. Homogeneous sets in the sense of Solymosi and Vu easily satisfy this condition, as do other sets that in the sense that will be made precise below respect the curvature properties of the sphere.

The classical Erdős distance conjecture (EDC) says that a planar point set of cardinality  $N$  determines at least a constant times  $\frac{N}{\sqrt{\log(N)}}$  Euclidean distances. Taking  $A = [0, \sqrt{N}]^2 \cap \mathbb{Z}^2$  shows that such an estimate would be best possible ([Erd46]). More precisely, let  $A \subset \mathbb{R}^2$  with  $\#A = N$ . Let

$$(0.1) \quad \Delta(A) = \{\|x - y\| : x, y \in A\},$$

where  $\|z\| = \sqrt{z_1^2 + z_2^2}$ . See, for example, [PA96] and [PS04] for a description of this beautiful problem and connections with other problems in geometric combinatorics.

In spite of many efforts over the past sixty years, the problem remains unsolved. The best known result to date, due to Katz and Tardos ([KT04]), partly based on an ingenious combinatorial technique by Solymosi and Tóth ([ST01]), gives

$$(0.2) \quad \#\Delta(A) \gtrsim N^{\approx .86}.$$

Here and throughout the paper,  $a \lesssim b$  ( $a \gtrsim b$ ) means that there exists a universal constant  $C$  such that  $a \leq Cb$  ( $a \geq Cb$ ) and  $a \approx b$  means that  $a \lesssim b$  and  $b \lesssim a$ .

Many of the aforementioned papers on the EDC observe that their method extends to distinct distances on spheres. This is partly due to the fact that many of the principles of planar geometry such as the existence of easily constructed geodesics and bisectors carries over to the spherical setting. In this paper we shall see that the curvature properties of the unit sphere allow for a very efficient accounting procedure for the distance set and lead to a complete resolution of the distance conjecture in this context, under an additional assumption on the set  $A$ , that in effect enables one to take advantage of curvature. Our main result is the following.

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**Theorem 0.1.** *Let  $A \subset S^2$  be a finite set of cardinality  $N$ . Let  $\Delta(A)$  be defined as above. Let  $\theta(a, b)$  denote the spherical, or angular, distance between  $a$  and  $b$ ,  $a, b \in S^2$ . Then*

$$(0.3) \quad \#\Delta(A) \gtrsim \frac{N}{I_1(A)},$$

where

$$(0.4) \quad I_\beta(N) = \frac{1}{N^2} \sum_{a \neq b} \frac{1}{\theta^\beta(a, b)}.$$

The applicability of Theorem 0.1 depends on the extent to which we can bound  $I_1(N)$  from above. For example, the quantity  $I_1(N)$  is unbounded, as  $N \rightarrow \infty$ , if  $A$  is a maximal  $\frac{1}{\sqrt{N}}$ -separated subset of  $S^2$  contained in a curved  $\frac{1}{\sqrt{N}}$  by  $\frac{1}{\sqrt{N}}$  rectangle. On the other hand,  $I_1(N)$  is bounded if  $A$  is  $\frac{1}{\sqrt{N}}$  separated. Observe that one can partition the sphere into  $N$   $\frac{1}{\sqrt{N}}$  by  $\frac{1}{\sqrt{N}}$  curved rectangles with the property that any point of  $S^2$  is covered by at most three of these rectangles. Thus this special case of boundedness of  $I_1(N)$  can be viewed as a spherical analog of the homogeneity condition on the point set employed by Solymosi and Vu ([SV03]). Also observe that if  $A$  consists of  $N$  equally spaced points on a great circle in  $S^2$ ,  $I_1(N)$  grows logarithmically in  $N$ .

We conclude that for a large class of point sets on  $S^2$ , the EDC holds. Moreover, we conjecture that on  $S^2$ , the Erdős Distance Conjecture should hold without the logarithmic factor as the lattice example from  $\mathbb{R}^2$  has no apparent analog in this setting.

Observe that  $I_\beta(A)$  is a discrete analog of the energy integral used to measure Hausdorff dimension of sets in a continuous setting. This connection is explored at the end of the paper.

The method of proof of Theorem 0.1 easily generalizes to higher dimensional spheres. In the process, interesting issues involving the geometric properties of the function  $I_1(A)$  arise. We shall investigate this matter systematically in a subsequent paper.

## PROOF OF THEOREM 0.1

Let  $a, b \in S^2$ . Then  $\|a - b\|^2 = \|a\|^2 + \|b\|^2 - 2a \cdot b = 2 - 2a \cdot b$ . It follows that instead of counting Euclidean distances on  $S^2$ , it suffices to count dot products  $a \cdot b$ . Let  $h_{a,\delta}$  be a smooth cutoff function on  $S^2$ , identically equal to 1 in the  $\delta$ -neighborhood of  $a \in S^2$ , and vanishing outside the  $2\delta$ -neighborhood of  $a$ . For convenience we construct this function in such a way that any  $h_{a,\delta}$  can be obtained from a  $h_{b,\delta}$  by a rotation that takes  $a$  to  $b$ . The right choice for  $\delta$  will turn out to be  $\frac{1}{N}$ , but we will keep this parameter flexible for a while for the sake of clarity.

**Construction of measures approximating  $A$  and  $\Delta(A)$ .** Let  $\omega \in S^2$  and

$$(1.1) \quad \mu(\omega) = c_1 \frac{1}{N\delta^2} \sum_{a \in A} h_{a,\delta}(\omega),$$

a measure approximating  $A$ . Clearly the constant  $c_1$  can be chosen such that  $\mu$  is a probability measure, so in the sequel assume  $c_1 = 1$ . We now construct a measure approximating  $\Delta(A)$ . Let  $A_\delta$  denote the support of  $\mu$ , the set where  $\mu$  is non-zero. Let  $f$  be a function on  $\Delta(A_\delta)$ . Then the distance measure  $\nu$  is defined via the following identity:

$$(1.2) \quad \int f(t) d\nu(t) = \int \int f(\|x - y\|) d\mu(x) d\mu(y).$$

In case of  $S^2$ , see above, when one looks at angular distances, instead of (1.2) one can use the following definition for the distance measure  $\nu$ :

$$(1.3) \quad \int f(t) d\nu(t) = \int \int f(x \cdot y) d\mu(x) d\mu(y).$$

Our plan is to show that  $\nu$  defined via (1.3) has an  $L^2$  density and then convert this statement into a lower bound for the number of distances. On the combinatorial level, what we are doing can be described as follows. Let  $m(t) = \#\{(x, y) \in A \times A : \|x - y\| = t\}$ , the incidence function. Estimating the  $L^2$  norm of  $\nu$  with an appropriate  $\delta$ , turns out to be equivalent to estimating  $\sum_{t \in \Delta(A)} m^2(t)$  from above. The latter is easily converted into a lower bound for the number of distances using the Cauchy-Schwartz inequality and the fact that  $\sum_{t \in \Delta(A)} m(t) \approx N^2$ .

**Estimation of the  $L^2$  norm of  $\nu$ .** Taking  $f(t) = e^{2\pi i \lambda t}$  in (1.3) we get

$$(1.4) \quad \widehat{\nu}(\lambda) = \frac{1}{N^2 \delta^4} \sum_{a, b \in A} \int_{S^2} \widehat{h_{a,\delta}}(\lambda \omega) h_{b,\delta}(\omega) d\omega.$$

By the standard method of stationary phase (see e.g. [W03]),

$$(1.5) \quad |\widehat{h_{a,\delta}}(\lambda \omega)| \lesssim \min\{\delta^2, \lambda^{-1}\},$$

if  $\omega \in U_{a,2\delta}$ , where  $U_{a,2\delta}$  is the  $2\delta$ -neighborhood of  $a$ , and

$$(1.6) \quad |\widehat{h_{a,\delta}}(\lambda \omega)| \lesssim \min\{\delta^2, \delta^2 [\lambda \theta(\omega, U_{a,\delta})]^{-M}\},$$

if  $\omega \notin U_{a,2\delta}$ , where  $\theta(\omega, U_{a,\delta})$  is the angular distance from  $\omega$  to  $U_{a,\delta}$ . The exponent  $M$  in (1.6) is an arbitrary positive integer,<sup>1</sup> but we shall confine ourselves to the case  $M = 2$ .

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<sup>1</sup>The formulas (1.5) and (1.6) are obtained as follows. If the direction  $\omega$  on the “Fourier side” coincides with one of the normal directions to the spherical cap in question, one gets the standard decay for the Fourier transform of Lebesgue measure on the unit sphere in (1.5). Otherwise, one can integrate by parts  $M$  times and using the trivial area bound  $\delta^2$  for the oscillatory integral in the final step.

Plugging (1.5) and (1.6) into (1.4) yields

$$\begin{aligned}
|\widehat{\nu}(\lambda)| &\lesssim \frac{1}{N} \chi_{[-\delta^{-2}, \delta^{-2}]}(\lambda) + \frac{1}{N\lambda\delta^2} \chi_{\mathbb{R} \setminus [-\delta^{-2}, \delta^{-2}]}(\lambda) \\
&+ \frac{1}{N^2} \sum_{a \neq b} \chi_{[-\theta^{-1}(a,b), \theta^{-1}(a,b)]}(\lambda) + \frac{1}{N^2} \sum_{a \neq b} \chi_{\mathbb{R} \setminus [-\theta^{-1}(a,b), \theta^{-1}(a,b)]}(\lambda) [\lambda \theta(a,b)]^{-2} \\
(1.7) \qquad &= I + II + III + IV,
\end{aligned}$$

where  $\chi_J$  is the characteristic function of a set  $J$ . Observe that the first line in (1.7) in essence corresponds to setting  $a = b$  in (1.4), when the estimate (1.5) comes into play, while the second line in (1.7) is the case  $a \neq b$  in (1.4), which uses the estimate (1.6).

A straightforward calculation shows that

$$(1.8) \qquad \int I^2 d\lambda + \int II^2 d\lambda \lesssim 1,$$

provided that

$$(1.9) \qquad \delta \gtrsim \frac{1}{N}.$$

We now estimate  $III$  and  $IV$ . We have

$$\begin{aligned}
\int III^2 d\lambda &= \frac{1}{N^4} \sum_{a \neq a'; b \neq b'} \chi_{[-\theta^{-1}(a,b), \theta^{-1}(a,b)]}(\lambda) \chi_{[-\theta^{-1}(a',b'), \theta^{-1}(a',b')]}(\lambda) d\lambda \\
(1.10) \qquad &\lesssim \frac{1}{N^2} \sum_{a \neq b} \theta^{-1}(a,b) = I_1(N).
\end{aligned}$$

by assumption. The estimate on  $IV$  is clearly the same.

**The lower bound on the cardinality of the distance set.** By construction we have that

$$(1.11) \qquad \int d\nu(t) = 1.$$

By Cauchy-Schwartz it follows that

$$(1.12) \qquad 1 \leq |\text{supp}(\nu)| \cdot \int \nu^2(t) dt = |\text{supp}(\nu)| \cdot \int |\widehat{\nu}(\lambda)|^2 d\lambda,$$

4

where the second equality follows by Plancherel. Note that  $|supp(\nu)|$  denotes the Lebesgue measure of the support of  $\nu$ .

By construction,  $supp(\nu)$  consists of intervals of length  $\approx \delta$ , so

$$(1.13) \quad \#\Delta(A) \gtrsim \frac{|supp(\nu)|}{\delta} \gtrsim \frac{N}{I_1(N)},$$

since we have chosen  $\delta \approx \frac{1}{N}$ .

This completes the proof of Theorem 0.1. We conclude the argument by pointing out that the quantity

$$(1.14) \quad I_1(N) \approx \int \frac{d\nu(t)}{t} = \int \int |x - y|^{-1} d\mu(x) d\mu(y),$$

by definition of  $\nu$ . The last integral is always finite if  $\mu$  uniformly approximates a measure on a set of Hausdorff dimension greater than one. This need not be the case, however.

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